# Dirichlet $L$ -functions

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October 4, 2024

# Content

# Reference

# Backgroumd

In this lecture, we will establish the analytic properties of Dirichlet $L$ -functions. Let $\chi$ be a Dirichlet character $\bmod q$ ; Based on Riemann' s proof of the analytic continuation of the zeta-function, we have

$$\pi^{-s/2}\Gamma(s/2)\,L(s,\chi)=\int_0^\infty\left(\sum_{n\geq 1}\chi(n)\,e^{-\pi n^2 t}\right)t^{s/2}\frac{dt}{t}$$

and consider the **twisted theta series**

$$\widetilde{\theta}(t)=\sum_{n\in\mathbf{Z}}\chi(n)\,e^{-\pi n^2 t}=\sum_{n\geq 1}\left(\chi(n)+\chi(-1)\,\chi(n)\right)e^{-\pi n^2 t}.$$

## Definition

We say that a Dirichlet character $\chi$ is odd if $\chi(-1) = -1$ and even if $\chi(-1) = 1$ .

The above twisted theta series is identically zero if $\chi$ is odd. This leads to the following ’ correct’ definition of the twisted theta series: for a Dirichlet character $\chi \pmod q$ , we define $\epsilon \in \{0,1\}$ via $\chi(-1) = (-1)^{\epsilon}$ and set

$$\widetilde{\theta}_{\chi}(t) = \sum_{n\in\mathbf{Z}} n^{\epsilon}\chi(n)\, e^{-\pi n^2 t}.$$

We leave it as an exercise to check the identity

$$\pi^{-(s+\epsilon)/2}\Gamma\left(\frac{s+\epsilon}{2}\right) L(s,\chi) = \int_0^{\infty} \widetilde{\theta}_{\chi}(t)\, t^{(s+\epsilon)/2}\frac{dt}{t}$$

with $\epsilon \in \{0,1\}$ as above.(the formula like $\zeta$-functions says)

Our goal for the rest of this chapter is to prove the 'modularity' of these twisted theta series. In the setting of the Riemann zeta function, the modularity is deduced from **Poisson summation**. Here we have the problem that the summands in

$$\sum_{n\in\mathbf{Z}}\chi(n)\,f(n)=\sum_{n\in\mathbf{Z}}\chi(n)\,e^{-\pi n^2 t}$$

are a priori not functions on $\mathbf{R}$ . The first thing we will see is that we can interpolate (primitive) characters defined on $\mathbf{Z}$ to smooth functions defined on $\mathbf{R}$ via the device of Gauss sums.

## Proposition

*If* $d \mid q$ *and given a Dirichlet character* $\chi^* \pmod d$ *,*

$$\chi(n) = \begin{cases} \chi^*(n) & \textit{if } (n,q)=1, \\ 0 & \textit{otherwise} , \end{cases}$$

## Proof.

Complete multiplicativity: If $(mn,q)=1$ then $\chi(mn)=\chi(m)\chi(n)$ by definition. Otherwise there is a prime $p$ such that $p \mid q$ and $p \mid mn$ , which, since $p$ is prime, implies that $p \mid m$ or $p \mid n$ . Hence $(m,q)=1$ or $(n,q)=1$ and $\chi(mn)=0=\chi(m)\chi(n)$ .

Periodicity: If $(m,q)=1$ , then $\chi(m)=\chi^*(m)$ . Then for all $n \equiv m \pmod q$ , we automatically have $n \equiv m \pmod d$ (since $d \mid q$ ) and $(n,q)=1$ . Hence $\chi(m)=\chi(n)$ . If $(m,q) \neq 1$ then for all $n \equiv m \pmod q$ , we also have $(n,q) \neq 1$ and hence $\chi(m)=\chi(n)$ .

**Definition**

Let $\chi$ be a Dirichlet character (mod $q$ ). We call $d$ a quasiperiod of $\chi$ if $\chi(m) = \chi(n)$ when $m \equiv n \pmod d$ and $(mn, q) = 1$ . The smallest quasiperiod of $\chi$ is called the conductor of $\chi$ . If $\chi$ has conductor $q$ , we say that $\chi$ is primitive.

The terminology ' primitive' comes from the observation that if $\chi$ is primitive then it can not be induced by any character of smaller conductor. we have
$\chi(m) = \chi(n) \Leftrightarrow \chi^*(m) = \chi^*(n) \Leftrightarrow m \equiv n \pmod d$.
Hence if $d < q, \chi$ is not primitive. A character that is not primitive is called imprimitive. We leave it as an exercise to show that any Dirichlet character is induced by a primitive character. More precisely

## Proposition

*Let $\chi$ be a Dirichlet character $\pmod{q}$ with conductor $d$ . Then $d \mid q$ and there exists a unique primitive character $\chi^*\pmod{d}$ that induces $\chi$*

*We have the following criteria for primitivity.*

## Proposition

*Let $\chi$ be a Dirichlet character (mod $q$ ). The following statements are equivalent:*

*(1) $\chi$ is primitive;*

*(2) If $d \mid q, d < q$ , there is $c \in \mathbf{Z}$ such that $c \equiv 1\pmod{d}, (c,q) = 1, \chi\left(c\right) \neq 1$ ;*

*(3) If $d \mid q, d < q$ , then for every $h \in \mathbf{Z}$ , we have*

$$\sum_{\substack{a=1 \\ a\equiv h(d)}}^{q} \chi\left(a\right) = 0$$

## Example

The two Dirichlet characters mod 4 are: the principal character

$$\chi_0(n) = \begin{cases} 1 & (n,q) = 1 \\ 0 & (n,1) \neq 1 \end{cases}$$

and the odd character

$$\chi_4(n) = \begin{cases} 1 & n \equiv 1\,(4) \\ -1 & n \equiv 3\,(4) \\ 0 & n \text{ even} \end{cases}$$

$$= \begin{cases} (-1)^{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Observe that $\chi_4$ is primitive, and $\chi_0$ is not.

## Example

The principle character

$$\chi_0(n) = \begin{cases} 1 & (n,q)=1 \\ 0 & (n,1)\neq 1 \end{cases}$$

is primitive iff $q=1$ . In this case, $\chi_0(n)=1$ for all $n\in\mathbf{Z}$ and $L(s,\chi_0)=\zeta(s)$ .

## Gauss sums

We will need Gauss sums

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a)\, e(a/q)$$

where $\chi$ is a Dirichlet character $(\bmod\ q)$ and we use the notation $e(x) = e^{2\pi i x}$. The reader can check that $\overline{\tau(\chi)} = \chi(-1)\,\tau(\bar{\chi})$. We will use the Gauss sum to interpolate a (primitive) character defined on $\mathbf{Z}$ to a smooth function on $\mathbf{R}$. The idea is that the Gauss sum can be thought of as an inner product of a multiplicative character $\chi(a)$ with an additive character $e(a/q)$, which should be reminiscent of the construction of the $\Gamma$-function and its role in regularizing the $\zeta$-function.

# Twisted Poisson summation formula

### Theorem

*Let $\chi$ be a primitive Dirichlet character (mod $q$ ). Then*

$$\chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^{q} \bar{\chi}(a)\, e(an/q)$$

*and* $|\tau(\chi)| = \sqrt{q}$ .

### Theorem (Twisted Poisson summation formula)

*Let $\chi$ be a primitive character $\bmod q$ and let $f \in \mathcal{S}(\mathbf{R})$ . Then*

$$\sum_{n\in\mathbf{Z}} \chi(n)\, f(n) = \frac{1}{\tau(\bar{\chi})} \sum_{n\in\mathbf{Z}} \bar{\chi}(n)\, \widehat{f}(n/q)\,.$$

**Proof.**

The Poisson summation formula can be applied to $g(n) := \chi(n) f(n)$, seen as a function on $\mathbf{R}$. We have

$$\widehat{g}(t) = \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^{q} \bar{\chi}(a) \int_{\mathbf{R}} f(x) e((t + a/q)x)\, dx = \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^{q} \bar{\chi}(a) \widehat{f}(t +$$

Then

$$\sum_{n\in\mathbf{Z}} g(n) = \sum_{n\in\mathbf{Z}} \widehat{g}(n) = \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^{q} \bar{\chi}(a) \sum_{n\in\mathbf{Z}} \widehat{f}((qn+a)/q)$$

$$= \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^{q} \bar{\chi}(qn+a) \sum_{n\in\mathbf{Z}} \widehat{f}((qn+a)/q)$$

**Remark**

Note that for $q=1$ and $\chi=\chi_0$ , we recover the usual Poisson summation formula.
Recall that for a Dirichlet character $\chi$ , we have introduced the twisted theta series

$$\widetilde{\theta}_\chi(t)=\sum_{n\in\mathbf{Z}} n^\epsilon \chi(n)\, e^{-\pi n^2 t}$$

where $\epsilon\in\{0,1\}$ is determined by $\chi(-1)=(-1)^\epsilon$ .

**Theorem**

*Let $\chi$ be a primitive character $\mod q$ and $t>0$ . Then*

$$\widetilde{\theta}_\chi(t)=\frac{(-i)^\epsilon \tau(\chi)}{q^{1+\epsilon}t^{1/2+\epsilon}}\widetilde{\theta}_{\bar{\chi}}\left(\frac{1}{q^2 t}\right).$$

**Proof.**

Apply the twisted Poisson summation formula to
$f(x) = x^{\epsilon} e^{-\pi x^2 t}$ $\square$

# Elements of the analytic theory of $L(s,\chi)$

**Theorem**

*Let $\chi$ be a primitive character $\bmod q, \chi \neq \chi_0$ . The function*

$$\Lambda(s,\chi) = \pi^{-(s+\epsilon)/2}\Gamma\left(\frac{s+\epsilon}{2}\right)L(s,\chi)$$

*admits a holomorphic continuation to all $s \in \mathbf{C}$ that satisfies the functional equation*

$$\Lambda(s,\chi) = (-i)^{\epsilon}\tau(\chi)\,q^{-s}\Lambda(1-s,\bar{\chi}).$$

*You cam compare with*

$$\xi(s) = \xi(1-s)$$

### Remark

Note that if $\chi = \chi_0$ , then $L(s,\chi) = \zeta(s)$ and

$$\sum_{n\geq 1} n^{\epsilon}\chi(n)\, e^{-\pi n^2 t} = \frac{\widetilde{\theta}_{\chi}(t) - 1}{2}.$$

one can see that the appearance of the simple poles at $s = 0, 1$ is due to the additive correction factor of $-1/2$.

The Riemann playbook may be applied to Dirichlet $L$ -functions.

First note that

$$L(s,\chi) = \exp\left(\sum_{p}\sum_{r\geq 1}\frac{\chi(p)}{rp^{rs}}\right)$$

# Generalized Riemann hypothesis

**The generalized Riemann hypothesis (GRH)** conjectures that if $\Lambda(s,\chi)=0$ then $\mathrm{Re}(s)=1/2$ . Let $(a,q)=1$ . We leave it to the reader to check that the following statements are equivalent:

(1) GRH

(2)

$$\sum_{\substack{n\leq x\\ n\equiv a(q)}}\Lambda(n)=\frac{x}{\varphi(q)}+O\left(\sqrt{x}(\log qx)^2\right)$$

(3)

$$\pi_{a(q)}(x)=\frac{\pi(x)}{\varphi(q)}+O\left(\sqrt{x}\log(qx)\right),$$

## Siegel-Walfisz,1936

where $\pi_{a(q)}(x)$ is the number of primes $p \leq x$ such that $p \equiv a\,(q)$ . A central theorem in the theory of Dirichlet $L$ -functions is the **Siegel-Walfisz theorem**, which can be seen as a refinement of both the prime number theorem and Dirichlet's theorem on arithmetic progressions.

**Theorem (Siegel-Walfisz,1936)**

*. Let $A > 0$ . There exists a constant $c > 0$ such that*

$$\pi_{a(q)}(x) = \frac{\pi(x)}{\varphi(q)} + O\left(xe^{-c\sqrt{\log x}}\right)$$

*for $(a,q) = 1$ and $q \leq (\log x)^A$ .*

**Remark**

The restriction on the range of $q$ is due to the possible existence of ' exceptional' (or Landau-Siegel) zeroes for quadratic characters. This is a new phenomenon with respect to the theory of the Riemann zeta function. Roughly we can say that there is a constant $c > 0$ such that $\{s = \sigma + it : \Lambda(s, \chi) = 0\} \cap \left(1 - \frac{c}{q|t|}, 1\right)$ if empty if $\chi$ is complex and contains at most one element if $\chi$ is quadratic, but we can not rule out that there indeed is a point in this interval

**Remark**

Let $q = 4$ . Then

$$\frac{\pi_{1(4)}(x)}{\pi(x)}, \frac{\pi_{3(4)}(x)}{\pi(x)} \sim \frac{1}{\varphi(4)} = \frac{1}{2}$$

THANK YOU!