Averages of Arithmetical Functions

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Chapter 1 Averages of Arithmetical Functions

1.1 Introduction

The average order of an arithmetic function f is defined as a simple real-valued function g such that

$$\sum_{n\leq x} f(n) \sim \sum_{n\leq x} g(n).$$

An arithmetic function f may possess multiple average orders. Our objective is to identify a function g that is straightforward in its expression and for which the asymptotic behavior is easily ascertainable. Notably, when faced with several potential mean orders, we consistently choose the one that minimizes the associated error.

Arithmetic functions often exhibit significant irregularities, complicating their study even with extensive numerical tables. For instance, the function $\tau(n)$, which takes values that oscillate between 2 (its value at each prime) and occasionally reaches considerably larger values, exemplifies this complexity. The process of averaging is crucial in mitigating these irregularities, effectively concealing occasional extreme deviations. Typically, particularly when f assumes positive values, the average order is non-trivial and represents the most straightforward quantity to determine an arithmetic function.

1.2 Euler's summation formula

Sometimes the asymptotic value of a partial sum can be obtained by comparing it with an integral. A summation formula of Euler gives an exact expression for the error made in such an approximation.

Theorem 1.1

Euler's summation formula. If f has a continuous derivative f' on the interval $[y, x]$,where $0 < y < x$,then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t - [t])f'(t)\,dt$$
$$+ f(x)([x] - x) - f(y)([y] - y).$$

♡

Proof Let $m = [y], k = [x]$ For integers n and $n-1$ in $[y, x]$ we have

$$\int_{n-1}^{n} [t]f'(t)\,dt = \int_{n-1}^{n} (n-1)f'(t)\,dt = (n-1)\{f(n) - f(n-1)\}$$
$$= \{nf(n) - (n-1)f(n-1)\} - f(n).$$

Summing from $n = m+2$ to $n = k$ we find the first sum telescopes, hence

$$\int_{m+1}^{k} [t]\,f'(t)\,dt = kf(k) - (m+1)f(m+1) - \sum_{n=m+2}^{k} f(n)$$
$$= kf(k) - mf(m+1) - \sum_{y<n\leq x} f(n).$$

Therefore

$$\sum_{y<n\leq x} f(n) = -\int_{m+1}^{k} [t]f'(t)\,dt + kf(k) - mf(m+1)$$
$$= -\int_{y}^{x} [t]f'(t)\,dt + kf(x) - mf(y).$$

Integration by parts gives us

$$\int_y^x f(t)\,dt = xf(x) - yf(y) - \int_y^x tf'(t)\,dt,$$

1.3 Some elementary asymptotic formulas

The next theorem gives a number of asymptotic formulas which are easy consequences of Euler's summation formula. $\zeta(s)$ denotes the Riemann zeta function which is defined by the equation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{if } s > 1,$$

Or

$$\zeta(s) = \lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) \text{ if } 0 < s < 1.$$

Theorem 1.2

If $x \ge 1$, we have:

$$\sum_{n\le x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right),$$

$$\sum_{n\le x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}) \quad \textit{if } s > 0, s \ne 1,$$

$$\sum_{n> x} \frac{1}{n^s} = O(x^{1-s}) \quad \textit{if } s > 1,$$

$$\sum_{n\le x} n^a = \frac{x^{a+1}}{a+1} + O(x^a) \quad \textit{if } a \ge 0.$$

♡

Proof

$$\begin{aligned}\sum_{n\le x} \frac{1}{n} &= \int_1^x \frac{dt}{t} - \int_1^x \frac{t-[t]}{t^2} dt + 1 - \frac{x-[x]}{x} \\ &= \log x - \int_1^x \frac{t-[t]}{t^2}\,dt \;+\; 1 \;+\; O\left(\frac{1}{x}\right) \\ &= \log x + 1 - \int_1^\infty \frac{t-[t]}{t^2} dt \;+\; \int_x^\infty \frac{t-[t]}{t^2}\,dt \;+\; O\left(\frac{1}{x}\right).\end{aligned}$$

The improper integral $\int_1^\infty (t-[t])\,t^{-2}\,dt$ dt exists sinceit isdominatedby $\int_1^\infty t^{-2}\,dt$ Also,

$$0 \le \int_x^\infty \frac{t-[t]}{t^2}\,dt \le \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x}$$

so the last equation becomes

$$\sum_{n\le x} \frac{1}{n} = \log x + 1 - \int_1^\infty \frac{t-[t]}{t^2}\,dt \;+\; O\left(\frac{1}{x}\right).$$

$$C = 1 - \int_1^\infty \frac{t-[t]}{t^2}\,dt.$$

Letting $x \to \infty$ in a we find that

$$\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n}-\log x\right)=1-\int_1^\infty \frac{t-[t]}{t^2}\,dt,$$

etc

1.4 The average order of $d(n)$

In this section we derive Dirichlet's asymptotic formula for the partial sums of the divisor function $d(n)$

Theorem 1.3

For all $x \ge 1$ we have

$$\sum_{n\le x} d(n) = x\log x + (2C-1)x + O(\sqrt{x}),$$

where C is Euler's constant

Proof

Since $d(n) = \sum_{d|n} 1$ we have

$$\sum_{n\le x} d(n) = \sum_{n\le x}\sum_{d|n} 1.$$

This is a double sum extended over n and d Since $d|n$ we can write $n = qd$ and extend the sum over all pairs of positive integers q, d d d with $qd \le x$ Thus

$$\sum_{n\le x} d(n) = \sum_{\substack{q,d\\ qd\le x}} 1.$$

lattice points on the horizontal line segment $1 \le q \le x/d$ andthen sum over all $d \le x$.

$$\sum_{n\le x} d(n) = \sum_{d\le x}\sum_{q\le x/d} 1.$$

Now we use 1.2 with $\alpha = 0$ to obtain

$$\sum_{q\le x/d} 1 = \frac{x}{d} + O(1).$$

Using this along with 1.2 we find

$$\sum_{n\le x} d(n) = \sum_{d\le x}\left\{\frac{x}{d}+O(1)\right\} = x\sum_{d\le x}\frac{1}{d}+O(x)$$
$$= x\left\{\log x + C + O\left(\frac{1}{x}\right)\right\} + O(x) = x\log x + O(x).$$

This is a weak version of 1.3which implies

$$\sum_{n\le x} d(n) \sim x\log x \quad \text{as } x\to\infty$$

and giveslogn as the average order of $d(n)$ To prove the more precise formula 1.3

counts the number of lattice points in a hyperbolic region and take advantage of the symmetry of the region about the line $q = d$. . The total number of lattice points in the region is equal to twice the number below the line $q = d$

we see that

$$\sum_{n\leq x} d(n) = 2\sum_{d\leq\sqrt{x}}\left\{\left[\frac{x}{d}\right]-d\right\}+[\sqrt{x}].$$

Now we use the relation $[y]=y+O(1)$ and 1.2 to obtain

$$\begin{aligned}\sum_{n\leq x} d(n) &= 2\sum_{d\leq\sqrt{x}}\left\{\frac{x}{d}-d+O(1)\right\}+O(\sqrt{x})\\ &= 2x\sum_{d\leq\sqrt{x}}\frac{1}{d}-2\sum_{d\leq\sqrt{x}}d+O(\sqrt{x})\\ &= 2x\left\{\log\sqrt{x}+C+O\left(\frac{1}{\sqrt{x}}\right)\right\}-2\left\{\frac{x}{2}+O(\sqrt{x})\right\}+O(\sqrt{x})\\ &= x\log x+(2C-1)x+O(\sqrt{x}).\end{aligned}$$

Note *The error term $O(\sqrt{x})$ can be improved.In 1903Voronoi proved that the error is $O(x^{1/3}\log x)$;in 1922van der Corput improved this to $O(x^{33/100})$ The best estimate to date is $O(x^{(1\,2/3\,7)+\varepsilon})$ for every $\varepsilon>0$, obtained by Kolesnik [35] in 1969. The determination of the infimum of all θ such that the error term is $O(x^{\theta})$ is an unsolved problem known as Dirichlet's divisor. problem. In 1915 Hardy and Landau showed that inf. $\theta\geq 1/4$*

[**b**]

1.5 The average order of the divisor functions $\sigma_x(n)$

The case $\alpha=0$ was considered in 1.3. Next we consider real $\alpha>0$ and treat the case $\alpha=1$ separately

Theorem 1.4

For all $x\geq 1$ we have

$$\sum_{n\leq x}\sigma_1(n)=\frac{1}{2}\zeta(2)x^2+O(x\log x).$$

Note *It can be shown that $\zeta(2)=\pi^2/6$. Therefore it shows that the average order of $\sigma_1(n)$ is $\pi^2 n/12$*

Proof

$$\begin{aligned}\sum_{n\leq x}\sigma_1(n) &= \sum_{n\leq x}\sum_{q|n}q=\sum_{\substack{q,d\\ qd\leq x}}q=\sum_{d\leq x}\sum_{q\leq x/d}q\\ &= \sum_{d\leq x}\left\{\frac{1}{2}\left(\frac{x}{d}\right)^2+O\left(\frac{x}{d}\right)\right\}\\ &= \frac{x^2}{2}\sum_{d\leq x}\frac{1}{d^2}+O\left(x\sum_{d\leq x}\frac{1}{d}\right)\\ &= \frac{x^2}{2}\left(-\frac{1}{x}+\zeta(2)+O\left(\frac{1}{x^2}\right)\right)+O(x\log x)\\ &= \frac{1}{2}\zeta(2)x^2+O(x\log x).\end{aligned}$$

Theorem 1.5

If $x\geq 1$ and α>0 $\alpha>0$ $\alpha>0,\alpha\neq 1$ α≠1 $\alpha\neq 1$,we have

$$\sum_{n\leq x}\sigma_\alpha(n)=\frac{\zeta(\alpha+1)}{\alpha+1}\,x^{\alpha+1}\;+\;O(x^{\beta}),$$

where $\beta = \max\{1, \alpha\}$ ♡

Proof

$$\begin{aligned}
\sum_{n \le x} \sigma_\alpha(n) &= \sum_{n \le x} \sum_{q|n} q^\alpha = \sum_{d \le x} \sum_{q \le x/d} q^\alpha \\
&= \sum_{d \le x} \left\{ \frac{1}{\alpha+1} \left(\frac{x}{d} \right)^{\alpha+1} + O\left(\frac{x^\alpha}{d^\alpha} \right) \right\} = \frac{x^{\alpha+1}}{\alpha+1} \sum_{d \le x} \frac{1}{d^{\alpha+1}} + O\left(x^\alpha \sum_{d \le x} \frac{1}{d^\alpha} \right) \\
&= \frac{x^{\alpha+1}}{\alpha+1} \left\{ \frac{x^{-\alpha}}{-\alpha} + \zeta(\alpha+1) + O(x^{-\alpha-1}) \right\} \\
&+ O\left(x^a \left\{ \frac{x^{1-a}}{1-\alpha} + \zeta(\alpha) + O(x^{-a}) \right\} \right) \\
&= \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x) + O(1) + O(x^\alpha) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^\beta)
\end{aligned}$$

where $\beta = \max\{1, \alpha\}$

To find the average order of $\sigma_a(n)$ for negative α wewrite $\alpha = -\beta$ where $\beta > 0$

Theorem 1.6

If $\beta > 0$ *let* $\delta = \max\{0, 1-\beta\}$ *Then if* $x > 1$ *we have*

$$\begin{aligned}
\sum_{n \le x} \sigma_{-\beta}(n) &= \zeta(\beta + 1)x + .O(x^\delta) \quad \text{if } \beta \ne 1, \\
&= \zeta(2)x + O(\log x) \quad if\ \beta = 1.
\end{aligned}$$

♡

Proof We have

$$\begin{aligned}
\sum_{n \le x} \sigma_{-\beta}(n) &= \sum_{n \le x} \sum_{d|n} \frac{1}{d^\beta} = \sum_{d \le x} \frac{1}{d^\beta} \sum_{q \le x/d} 1 \\
&= \sum_{d \le x} \frac{1}{d^\beta} \left\{ \frac{x}{d} + O(1) \right\} = x \sum_{d \le x} \frac{1}{d^{\beta+1}} + O\left(\sum_{d \le x} \frac{1}{d^\beta} \right).
\end{aligned}$$

The last term is $O(\log x)$ if $\beta = 1$ and $O(x^\delta)$ if $\beta \ne 1$ .Since

$$x \sum_{d \le x} \frac{1}{d^{\beta+1}} = \frac{x^{1-\beta}}{-\beta} + \zeta(\beta+1)x + O(x^{-\beta}) = \zeta(\beta+1)x + O(x^{1-\beta})$$

1.6 The average order of $\varphi(n)$

The asymptotic formula for the partial sums of Euler's totient involves the sum of the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}.$$

This series converges absolutely since it is dominatedby $\sum_{n=1}^{\infty} n^{-2}$. In a later chapter we will prove that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Assuming this result for the time being we have.

$$\sum_{n\le x}\frac{\mu(n)}{n^2}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}-\sum_{n>x}\frac{\mu(n)}{n^2}$$
$$=\frac{6}{\pi^2}+O\left(\sum_{n>x}\frac{1}{n^2}\right)=\frac{6}{\pi^2}+O\left(\frac{1}{x}\right)$$

Theorem 1.7

For $x>1$ we have

$$\sum_{n\le x}\varphi(n)=\frac{3}{\pi^2}\,x^2+O(x\log x),$$

so the average order of $\varphi(n)$ is $3n/\pi^2$

Proof The method is similar to that used for the divisor functions.We start with the relation

$$\varphi(n)=\sum_{d\mid n}\mu(d)\,\frac{n}{d}$$

and obtain

$$\sum_{n\le x}\varphi(n)=\sum_{n\le x}\sum_{d\mid n}\mu(d)\,\frac{n}{d}=\sum_{\substack{q,d\\ qd\le x}}\mu(d)q=\sum_{d\le x}\mu(d)\sum_{q\le x/d}q$$
$$=\sum_{d\le x}\mu(d)\left\{\frac{1}{2}\left(\frac{x}{d}\right)^2+O\left(\frac{x}{d}\right)\right\}$$
$$=\frac{1}{2}\,x^2\sum_{d\le x}\frac{\mu(d)}{d^2}+O\left(x\sum_{d\le x}\frac{1}{d}\right)$$
$$=\frac{1}{2}\,x^2\left\{\frac{6}{\pi^2}+O\left(\frac{1}{x}\right)\right\}+O(x\,\log x)=\frac{3}{\pi^2}\,x^2+O(x\,\log x).$$

1.7 The average order of $\mu(n)$ and of $\Lambda(n)$

The average orders of $\mu(n)$ and $\Lambda(n)$ are considerably more difficult to determine than those of $\varphi(n)$ and the divisor functions.It is known that $\mu(n)$ has average order O and that $\Lambda(n)$ has average order1.That is,

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\mu(n)=0$$

and

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\Lambda(n)=1,$$

but the proofs are not simple.Both these results are equivalent to the prime number theorem (you can see my article ——PNT)

$$\lim_{x\to\infty}\frac{\pi(x)\log x}{x}=1,$$

1.8 The partial sums of a Dirichlet product

Theorem 1.8

If $h = f * g$ *,let*

$$H(x) = \sum_{n \le x} h(n), \quad F(x) = \sum_{n \le x} f(n), \quad and\ G(x) = \sum_{n \le x} g(n).$$

Then we have

$$H(x) = \sum_{n \le x} f(n) G\left(\frac{x}{n}\right) = \sum_{n \le x} g(n) F\left(\frac{x}{n}\right).$$

♡

Proof

$$U(x) = \begin{cases} 0 & \text{if} 0 < x < 1, \\ 1 & \text{if} x \ge 1. \end{cases}$$

Then $F = f \circ U$ $G = g \circ U$, and we have

$$f \circ G = f \circ (g \circ U) = (f * g) \circ U = H, g \circ F = g \circ (f \circ U) = (g * f) \circ U = H.$$

If $g(n) = 1$ for all n then $G(x) = [x]$ gives us the following corollary:

Corollary 1.1

If $F(x) = \sum_{n \le x} f(n)$ *we have*

$$\sum_{n \le x} \sum_{d \mid n} f(d) = \sum_{n \le x} f(n) \left[\frac{x}{n}\right] = \sum_{n \le x} F\left(\frac{x}{n}\right).$$

♡

1.9 Applications to $\mu(n)$ and $\Lambda(n)$

Now we take $f(n) = \mu(n)$ and $\Lambda(n)$ will be used later in studying the distribution of primes

Theorem 1.9

For $x \ge 1$ *we have*

$$\sum_{n \le x} \mu(n) \left[\frac{x}{n}\right] = 1$$

and

$$\sum_{n \le x} \Lambda(n) \left[\frac{x}{n}\right] = \log[x]!.$$

♡

Proof we have

$$\sum_{n \le x} \mu(n) \left[\frac{x}{n}\right] = \sum_{n \le x} \sum_{d \mid n} \mu(d) = \sum_{n \le x} \left[\frac{1}{n}\right] = 1$$

and

$$\sum_{n\le x}\Lambda(n)\left[\frac{x}{n}\right]=\sum_{n\le x}\sum_{d|n}\Lambda(d)=\sum_{n\le x}\log n=\log[x]!.$$

Note

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}$$

converges and has a sum 0.

Theorem 1.10

For all $x\ge 1$ we have

$$\left|\sum_{n\le x}\frac{\mu(n)}{n}\right|\le 1,$$

with equality holding only if. $x<2$

Proof If $x<2$ there is only one term in the sum, $\mu(1)=1$ Now assume that $x\ge 2$ For each real y let $\{y\}=y-[y]$ .Then

$$1=\sum_{n\le x}\mu(n)\left[\frac{x}{n}\right]=\sum_{n\le x}\mu(n)\left(\frac{x}{n}-\left\{\frac{x}{n}\right\}\right)=x\sum_{n\le x}\frac{\mu(n)}{n}-\sum_{n\le x}\mu(n)\left\{\frac{x}{n}\right\}.$$

Since $0\le\{y\}<1$ this implies

$$\begin{aligned}x\left|\sum_{n\le x}\frac{\mu(n)}{n}\right|&=\left|1+\sum_{n\le x}\mu(n)\left\{\frac{x}{n}\right\}\right|\le 1+\sum_{n\le x}\left\{\frac{x}{n}\right\}\\&=1+\{x\}+\sum_{2\le n\le x}\left\{\frac{x}{n}\right\}<1+\{x\}+[x]-1=x.\end{aligned}$$

Dividing by x. We turn next to identity 1.9

$$\sum_{n\le x}\Lambda(n)\left[\frac{x}{n}\right]=\log[x]\,!,$$

and use it to determine the power of a prime which divides a factorial

Theorem 1.11 (Legendre's identity)

For every $x\ge 1$ we have

$$[x]!=\prod_{p\le x}p^{\alpha(p)}$$

where the product is extended over all primes $\le x$ and

$$\alpha(p)=\sum_{m=1}^{\infty}\left[\frac{x}{p^m}\right].$$

Note *The sum for $\alpha(p)$ is finite since $[x/p^m]=0$ for $p>x$*

Proof Since $\Lambda(n)=0$ unless n is a prime power,and $\Lambda(p^m)=\log p$,we have

$$\log[x]! = \sum_{n\le x}\Lambda(n)\left[\frac{x}{n}\right] = \sum_{p\le x}\sum_{m=1}^{\infty}\left[\frac{x}{p^m}\right]\ \log\ p\ = \sum_{p\le x}\alpha(p)\ \log\ p,$$

Next, we use Euler's summation formula to determine an asymptotic formula for $\log[x]!$

Theorem 1.12

If $x\ge 2$ we have

$$\log[x]! = x\log x - x + O(\log x),$$

and hence

$$\sum_{n\le x}\Lambda(n)\left[\frac{x}{n}\right] = x\log x - x + O(\log x).$$

♡

Proof

Taking $f(t)=\log t$ in Euler's summation formula we obtain

$$\begin{aligned}\sum_{n\le x}\log n &= \int_1^x \log t\,dt\ +\int_1^x \frac{t-[t]}{t}\,dt\ -(x\ -\ [x])\log x\\ &= x\log x - x + 1 + \int_1^x \frac{t-[t]}{t}\,dt\ +\ O(\log x).\end{aligned}$$

since

$$\int_1^x \frac{t-[t]}{t}\,dt = O\left(\int_1^x \frac{1}{t}\,dt\right) = O(\log x),$$

Theorem 1.13

For $x\ge 2$ we have

$$\sum_{p\le x}\left[\frac{x}{p}\right]\log p = x\log x + O(x),$$

where the sum is extended over all primes $\le x$

♡

Proof Since $\Lambda(n)=0$ unless n is a prime power we have

$$\sum_{n\le x}\left[\frac{x}{n}\right]\Lambda(n) = \sum_{p}\sum_{m=1}^{\infty}\left[\frac{x}{p^m}\right]\Lambda(p^m).$$

Now $p^m\le x$ implies $p\le x$ .Also, $[x/p^m]=0$ if $p>x$ so we can write the last sum as

$$\sum_{p\le x}\sum_{m=1}^{\infty}\left[\frac{x}{p^m}\right]\log p\ =\ \sum_{p\le x}\left[\frac{x}{p}\right]\log p\ +\ \sum_{p\le x}\sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right]\log p.$$

Next we prove that the last sum is $O(x)$ we have

$$\begin{aligned}\sum_{p\le x}\log p\sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right] &\le \sum_{p\le x}\log p\sum_{m=2}^{\infty}\frac{x}{p^m} = x\sum_{p\le x}\log p\sum_{m=2}^{\infty}\left(\frac{1}{p}\right)^m \\ &= x\sum_{p\le x}\log p\cdot\frac{1}{p^2}\cdot\frac{1}{1-\frac{1}{p}} = x\sum_{p\le x}\frac{\log p}{p(p-1)} \\ &\le x\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)} = O(x).\end{aligned}$$

Hence we have shown that

$$\sum_{n\le x}\left[\frac{x}{n}\right]\Lambda(n) = \sum_{p\le x}\left[\frac{x}{p}\right]\ \log\ p\ +\ O(x),$$

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