Dirichlet L -functions

Author: Jiahai Wang

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Chapter 1 Dirichlet L -functions

The theory of analytic functions has many applications in number theory. A particularly spectacular application was discovered by Dirichlet who proved in 1837 that there are infinitely many primes in any arithmetic progression $b, b+m, b+2m, \ldots$, where $(m, b)=1$. To do this he introduced the L-functions which bear his name. In this chapter we will define these functions, investigate their properties, and prove the theorem on arithmetic progressions. The use of Dirichlet L-functions extends beyond the proof of this theorem. It turns out that their values at negative integers are especially important. We will derive these values and show how they relate to Bernoulli numbers.

1.1 Zeta Function

The Riemann zeta function $\zeta(s)$ is defined by $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$. It converges for $s>1$ and converges uniformly for $s \geq 1+\delta>1$, for each $\delta>0$.

Proposition 1.1

For $s>1$

$$\zeta(s)=\prod_{p}\left(1-p^{-s}\right)^{-1}$$

where the product is over all primes $p>0$

Proof For $s>1, p^{-s}<1$, so we have $\left(1-p^{-s}\right)^{-1}=\sum_{m=0}^{\infty} p^{-ms}$.By the theorem of unique factorization

$$\prod_{p \leq N}\left(1-p^{-s}\right)^{-1}=\sum_{n \leq N} n^{-s}+R_{N}(s)$$

Clearly, $R_{N}(s) \leq \sum_{n=N+1}^{\infty} n^{-s}$. Since $\zeta(s)$ converges, $R_{N}(s) \rightarrow 0$ as $N \rightarrow \infty$. The result follows.

The behavior of $\zeta(s)$ as $s \rightarrow 1$ is very important. Since $\sum_{n=1}^{\infty} n^{-1}$ diverges we, of course, suspect $\zeta(s) \rightarrow \infty$ as $s \rightarrow 1$. In fact,

Proposition 1.2

Assume $s>1$. Then

$$\lim_{s \rightarrow 1}(s-1) \zeta(s)=1$$

Proof For fixed s, t^{-s} is a monotone decreasing function of t . Thus,

$$(n+1)^{-s}<\int_{n}^{n+1} t^{-s} dt<n^{-s}.$$

Summing from $n=1$ to ∞ ,

$$\zeta(s)-1<\int_{1}^{\infty} t^{-s} dt<\zeta(s)$$

The value of the integral is $(s-1)^{-1}$. It follows that $1<(s-1) \zeta(s)<s$. Taking the limit as $s \rightarrow 1$ gives the result.

Corollary 1.1

As $s \to 1$ we have

$$\frac{\ln \zeta(s)}{\ln (s-1)^{-1}} \to 1$$

Proof Let $(s-1)\zeta(s) = \rho(s)$. Then $\ln(s-1) + \ln \zeta(s) = \ln \rho(s)$ so we have $\ln \zeta(s) / \ln (s-1)^{-1} = 1 + \left(\ln \rho(s) / \ln (s-1)^{-1}\right)$.

As $s \to 1, \rho(s) \to 1$ by the proposition. Therefore, $\ln \rho(s) \to 0$ and the result follows.

Proposition 1.3

$\ln \zeta(s) = \sum_p p^{-s} + R(s)$ *where* $R(s)$ *remains bounded as* $s \to 1$.

Proof We use the formula $-\ln(1-x) = x + x^2/2 + x^3/3 + \cdots$ which is valid for $-1 < x < 1$.

we have

$$\zeta(s) = \prod_{p \le N} \left(1 - p^{-s}\right)^{-1} \lambda_N(s)$$

where $\lambda_N(s) \to 1$ as $N \to \infty$. Taking the logarithm of both sides yields $\ln \zeta(s) = \sum_{p \le N} \sum_{m=1}^{\infty} m^{-1} p^{-ms} + \ln \lambda_N(s)$.

Taking the limit as $N \to \infty$

$$\begin{aligned} \ln \zeta(s) &= \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-ms} \\ &= \sum_p p^{-s} + \sum_p \sum_{m=2}^{\infty} m^{-1} p^{-ms}. \end{aligned}$$

The second sum is less than

$$\begin{aligned} \sum_p \sum_{m=2}^{\infty} p^{-ms} &= \sum_p p^{-2s} \left(1 - p^{-s}\right)^{-1} \\ &\le \left(1 - 2^{-s}\right)^{-1} \sum_p p^{-2s} \le 2\zeta(2). \end{aligned}$$

Throughout we have used the assumption that $s > 1$.

Definition 1.1

A set of positive primes $\mathcal{P}$ *is said to have Dirichlet density if*

$$\lim_{s \to 1} \frac{\sum_{p \in \mathcal{P}} p^{-s}}{\ln (s-1)^{-1}}$$

exists. If the limit exists we set it equal to $\mathrm{d}(\mathcal{P})$ *and call* $\mathrm{d}(\mathcal{P})$ *the Dirichlet density of* $\mathcal{P}$.

Proposition 1.4

Let $\mathcal{P}$ *be a set of positive prime numbers. Then*

(a) If $\mathcal{P}$ *is finite, then* $\mathrm{d}(\mathcal{P}) = 0$.

(b) If $\mathcal{P}$ *consists of all but finitely many positive primes, then* $\mathrm{d}(\mathcal{P}) = 1$.

(c) If $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ *where* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are disjoint and* $\mathrm{d}(\mathcal{P}_1)$ *and* $\mathrm{d}(\mathcal{P}_2)$ *both exist, then* $\mathrm{d}(\mathcal{P}) = \mathrm{d}(\mathcal{P}_1) + \mathrm{d}(\mathcal{P}_2)$.

Proof Parts (a) and (c) are clear from the definition of Dirichlet density. Part (b) follows quickly from the corollary to Proposition 16.1.2 and Proposition 16.1.3.

We are now in a position to state the main theorem of this chapter. The proof will be spread out over the next three sections.

Theorem 1.1 (L. Dirichlet)

Suppose $a, m \in \mathbb{Z}$, with $(a, m) = 1$. Let $\mathcal{P}(a; m)$ be the set of positive primes p such that $p \equiv a\,(m)$. Then $d(\mathcal{P}(a; m)) = 1/\phi(m)$. ♡

Note *The Theorem certainly implies $\mathcal{P}(a; m)$ is infinite, since if it were finite its density would be zero.*

1.2 Dirichlet Characters

omit

1.3 Dirichlet L -functions

Definition 1.2

Let χ be a Dirichlet character modulo m.Dirichlet L-function associated to χ by the formula

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

♣

Since $|\chi(n) n^{-s}| \leq n^{-s}$ we see that the terms of $L(s, \chi)$ are dominated in absolute value by the corresponding terms of $\zeta(s)$. Thus $L(s, \chi)$ converges and is continuous for $s > 1$. Moreover, since χ is completely multiplicative we have a product formula for $L(s, \chi)$ in exactly the same way as for $\zeta(s)$. Namely,

$$L(s, \chi) = \prod_{p} \left(1 - \chi(p) p^{-s}\right)^{-1}.$$

Since $\chi(p) = 0$ for $p \mid m$ the above product is over positive primes not dividing m. The formula is valid for $s > 1$. There is a close connection between $L(s, \chi_0)$ and $\zeta(s)$. In fact,

$$L(s, \chi_0) = \prod_{p \nmid m} \left(1 - p^{-s}\right)^{-1}$$

$$= \prod_{p \mid m} \left(1 - p^{-s}\right) \prod_{p} \left(1 - p^{-s}\right)^{-1}$$

$$= \prod_{p \mid m} \left(1 - p^{-s}\right) \zeta(s)$$

From Proposition 1.1 we see $\lim_{s \to 1} (s - 1) L(s, \chi_0) = \prod_{p \mid m} \left(1 - p^{-1}\right) = \phi(m)/m$. In particular $L(s, \chi_0) \to \infty$ as $s \to 1$.

Let χ be a Dirichlet character and define $G(s, \chi) = \sum_{p} \sum_{k=1} (1/k) \chi\left(p^k\right) p^{-ks}$. Since $\left|(1/k) \chi\left(p^k\right) p^{-ks}\right| \leq p^{-ks}$ and since $\zeta(s)$ converges for $s > 1$ and converges uniformly for $s \geq 1 + \delta > 1$ we can conclude the same assertions are true for $G(s, \chi)$. Consequently $G(s, \chi)$ is continuous for $s > 1$. Moreover, for z a complex number with $|z| < 1$ we have $\exp\left(\sum_{k=1}^{\infty} (1/k) z^k\right) = (1 - z)^{-1}$, where exp denotes the usual exponential function. Substituting $z = \chi(p) p^{-s}$

we find $\exp\left(\sum_{k=1}^{\infty}(1/k)\,\chi\left(p^k\right)p^{-ks}\right)=\left(1-\chi\left(p\right)p^{-s}\right)^{-1}$ and a simple argument then shows $\exp G\left(s,\chi\right)=L\left(s,\chi\right)$ for all $s>1$. Thus the infinite series $G\left(s,\chi\right)$ provides an unambiguous definition for $\ln L\left(s,\chi\right)$. To avoid confusion we work directly with $G\left(s,\chi\right)$.

From the definition and the argument used in the proof of Proposition 16.1.3 we find

$$G\left(s,\chi\right)=\sum_{p\nmid m}\chi\left(p\right)p^{-s}+R_{\chi}\left(s\right) \tag{i}$$

where $R_{\chi}\left(s\right)$ remains bounded as $s\to 1$. Multiply both sides of (i) by $\overline{\chi\left(a\right)}$ where $a\in\mathbb{Z},\left(a,m\right)=1$. Then sum over all Dirichlet characters modulo m . The result is

$$\sum_{\chi}\overline{\chi\left(a\right)}G\left(s,\chi\right)=\sum_{p\nmid m}p^{-s}\sum_{\chi}\overline{\chi\left(a\right)}\chi\left(p\right)+\sum_{\chi}\overline{\chi\left(a\right)}R_{\chi}\left(s\right).$$

thus

$$\sum_{\chi}\overline{\chi\left(a\right)}G\left(s,\chi\right)=\phi\left(m\right)\sum_{p\equiv a(m)}p^{-s}+R_{\chi,a}\left(s\right) \tag{ii}$$

where $R_{\chi,a}\left(s\right)$ remains bounded as $s\to 1$.

To conclude the proof of Theorem 1.1 we need the following proposition.

Proposition 1.5

If χ_0 denotes the trivial character modulo m , then $\lim_{s\to 1}G\left(s,\chi_0\right)/\ln\left(s-1\right)^{-1}=1$. If χ is a nontrivial Dirichlet character modulo m , then $G\left(s,\chi\right)$ remains bounded as $s\to 1$.

Proof The first assertion is easy. $L\left(s,\chi_0\right)$ is a real valued function of positive real numbers. We have seen $L\left(s,\chi_0\right)=\prod_{p|m}\left(1-p^{-s}\right)\zeta\left(s\right)$. It follows that $G\left(s,\chi_0\right)=\sum_{p|m}\ln\left(1-p^{-s}\right)+\ln\zeta\left(s\right)$.

The second assertion is quite deep. It is the most difficult part of the proof of Dirichlet's theorem on arithmetic progressions. We postpone the proof to the next section.

Now, assuming the above proposition, the proof of Dirichlet's theorem follows quickly from Equation (ii). We simply divide all the terms on both sides by $\ln\left(s-1\right)^{-1}$ and take the limit as $s\to 1$. By the above proposition, the limit on the left-hand side is 1 whereas the limit on the right-hand side is $\phi\left(m\right)d\left(\mathcal{P}\left(a;m\right)\right)$. Thus$d\left(\mathcal{P}\left(a;m\right)\right)=1/\phi\left(m\right)$ and we are done.

1.4 The Key Step

Up to now all our functions have been defined for $s>1$. We will show how to extend the domain of definition to $s>0$. In particular, if χ is nontrivial we will see that $L\left(1,\chi\right)$ is a well defined complex number and prove that $L\left(1,\chi\right)\neq 0$.

In what follows we will consider s as a complex variable. Write $s=\sigma+it$ where σ and t are real. The symbol σ will be used throughout to denote the real part of s .

If $a>0$ is real then $|a^s|=a^{\sigma}$. From this observation we see that the series defining $\zeta\left(s\right)$ and $L\left(s,\chi\right)$ converge and define an analytic function of the complex variable s in the half plane $\{s\in\mathbb{C}\mid\sigma>1\}$.

Lemma 1.1

Suppose $\{a_n\}$ and $\{b_n\}$ for $n=1,2,3,\ldots$ are sequences of complex numbers such that $\sum_{n=1}^{\infty}a_nb_n$ converges. Let $A_n=a_1+a_2+\cdots+a_n$ and suppose $A_nb_n\to 0$ as $n\to\infty$. Then

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} A_n (b_n - b_{n+1})$$

Proof Let $S_N = \sum_{n=1}^{N} a_n b_n$. Set $A_0 = 0$. Then

$$\begin{aligned} S_N = \sum_{n=1}^{N} (A_n - A_{n-1}) b_n &= \sum_{n=1}^{N} A_n b_n - \sum_{n=1}^{N} A_{n-1} b_n \\ &= \sum_{n=1}^{N} A_n b_n - \sum_{n=1}^{N-1} A_n b_{n+1} \\ &= A_N b_N + \sum_{n=1}^{N-1} A_n (b_n - b_{n+1}) . \end{aligned}$$

Taking the limit as $N \to \infty$ yields the result.

Proposition 1.6

$\zeta(s) - (s-1)^{-1}$ can be continued to an analytic function on the region $\{s \in \mathbb{C} \mid \sigma > 0\}$.

Proof Assume $\sigma > 1$. Then, by the lemma

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n \left(n^{-s} - (n+1)^{-s} \right) .$$

For a real number x recall that $[x]$ is the greatest integer less than or equal to x and $\langle x \rangle = x - [x]$. From the above expression for $\zeta(s)$ we find

$$\begin{aligned} \zeta(s) &= s \sum_{n=1}^{\infty} n \int_{n}^{n+1} x^{-s-1} dx \\ &= s \sum_{n=1}^{\infty} \int_{n}^{n+1} [x] x^{-s-1} dx \\ &= s \int_{1}^{\infty} [x] x^{-s-1} dx \\ &= s \int_{1}^{\infty} x^{-s} dx - s \int_{1}^{\infty} \langle x \rangle x^{-s-1} dx \\ &= \frac{s}{s-1} - s \int_{1}^{\infty} \langle x \rangle x^{-s-1} dx \end{aligned}$$

Since $|\langle x \rangle| \leq 1$ for all x the last integral converges and defines an analytic function for $\sigma > 0$. The result follows. We will use the same technique to extend $L(s, \chi)$ but first we need another lemma.

Lemma 1.2

Let χ be a nontrivial character modulo m . For all $N > 0$ we have $\left| \sum_{n=0}^{N} \chi(n) \right| \leq \phi(m)$.

Proof Write $N = qm + r$ where $0 \leq r < m$. Since $\chi(n+m) = \chi(n)$ for all n we see

$$\sum_{n=1}^{N} \chi(n) = q \left(\sum_{n=0}^{m-1} \chi(n) \right) + \sum_{n=0}^{r} \chi(n)$$

we have $\sum_{n=0}^{m-1} \chi(n) = 0$. Thus,

$$\left|\sum_{n=0}^{N} \chi(n)\right| = \left|\sum_{n=0}^{r} \chi(n)\right| \leq \sum_{n=0}^{m-1} |\chi(n)| = \phi(m) .$$

Proposition 1.7

Let χ be a nontrivial Dirichlet character modulo m. Then, $L(s, \chi)$ can be continued to an analytic function in the region $\{s \in \mathbb{C} \mid \sigma > 0\}$.

Proof Define $S(x) = \sum_{n \leq x} \chi(n)$.

By Lemma 1.1 we have for $\sigma > 1$,

$$\begin{aligned} L(s, \chi) &= \sum_{n=1}^{\infty} S(n) \left(n^{-s} - (n+1)^{-s}\right) \\ &= s \sum_{n=1}^{\infty} S(n) \int_{n}^{n+1} x^{-s-1} dx \\ &= s \int_{1}^{\infty} S(x) x^{-s-1} dx \end{aligned}$$

By Lemma 1.2, $|S(x)| \leq \phi(m)$ for all x . It follows that the above integral converges and defines an analytic function for all s such that $\sigma > 0$.

Our goal is to show that for χ nontrivial $L(1, \chi) \neq 0$. The next proposition will enable us to give a simple proof in the case where χ is a complex character, i.e., a character which takes on nonreal values.

Proposition 1.8

Let $F(s) = \prod_{x} L(s, \chi)$ where the product is over all Dirichlet characters modulo m . Then, for s real and $s > 1$ we have $F(s) \geq 1$.

Proof Assume s is real and $s > 1$. Recall that

$$G(s, \chi) = \sum_{p} \sum_{k=1}^{\infty} \frac{1}{k} \chi\left(p^{k}\right) p^{-ks}.$$

Summing over χ , we find

$$\sum_{\chi} G(s, \chi) = \phi(m) \sum \frac{1}{k} p^{-ks}$$

where the sum is over all primes p and integers k such that $p^{k} \equiv 1\,(m)$.

The right-hand side of the above equation is nonnegative (in fact, it is positive). Taking the exponential of both sides shows $\prod_{x} L(s, \chi) \geq 1$ as asserted.

Proposition 1.9

If χ is a nontrivial complex character modulo m , then $L(1, \chi) \neq 0$.

Proof From the series defining $L(s, \chi)$ we see that for s real, $s > 1$, $\overline{L(s, \chi)} = L(s, \bar{\chi})$. Letting s tend towards 1 it follows that $L(1, \chi) = 0$ implies $L(1, \bar{\chi}) = 0$.

Assume $L(1, \chi) = 0$ where χ is a complex character. The functions $L(s, \chi)$ and $L(s, \bar{\chi})$ are distinct and both have a zero at $s = 1$. In the product $F(s) = \prod_{\chi} L(s, \chi)$ we know $L(s, \chi_0)$ has a simple pole at $s = 1$ and all the other factors are analytic about $s=1$. It follows that $F(1)=0$. However, Proposition 1.8 shows $F(s) \geq 1$ for all real $s>1$. This is a contradiction. Therefore, $L(1, \chi) \neq 0$

Lemma 1.3

Suppose f is a nonnegative, multiplicative function on $\mathbb{Z}^{+}$, i.e., for all $m, n>0$ with $(m, n)=1, f^{\prime}(m n)=f(m) f(n)$. Assume there is a constant c such that $f\left(p^{k}\right)<c$ for all prime powers p^{k} . Then $\sum_{n=1}^{\infty} f(n) n^{-s}$ converges for all real $s>1$. Moreover

$$
\sum_{n=1}^{\infty} f(n) n^{-s}=\prod_{p}\left(1+\sum_{k=1}^{\infty} f\left(p^{k}\right) p^{-k s}\right) .
$$

Proof Fix $s>1$. Let $a(p)=\sum_{k=1}^{\infty} f\left(p^{k}\right) p^{-k s}$. Then $a(p)<c p^{-s} \sum_{k=0}^{\infty} p^{-k s}=c p^{-s}\left(1-p^{-s}\right)^{-1}$, and so $a(p)<2 c p^{-s}$. For positive x one has $1+x<\exp x$. Thus

$$
\prod_{p \leq N}(1+a(p))<\prod_{p \leq N} \exp a(p)=\exp \sum_{p \leq N} a(p) .
$$

Now, $\sum_{p \leq N} a(p)<2 c \sum_{p} p^{-s}=M$. From the definition of $a(p)$ and the multiplicativity of f we see $\sum_{n=1}^{N} f(n) n^{-s}<\prod_{p \leq N}(1+a(p))$. It follows that $\sum_{n=1}^{N} f(n) n^{-s}<\exp M$ for all N . Since f is, by assumption, nonnegative we have $\sum_{n=1}^{\infty} f(n) n^{-s}$ converges.

Theorem 1.2

Let χ be a nontrivial Dirichlet character modulo m. Then $L(1, \chi) \neq 0$.

Proof Having already proved that $L(1, \chi) \neq 0$ if χ is complex we assume χ is real.

Assume $L(1, \chi)=0$ and consider the function

$$
\psi(s)=\frac{L(s, \chi) L\left(s, \chi_{0}\right)}{L\left(2 s, \chi_{0}\right)}
$$

The zero of $L(s, \chi)$ at $s=1$ cancels the simple pole of $L\left(s, \chi_{0}\right)$ so the numerator is analytic on $\sigma>0$. The denominator is nonzero and analytic for $\sigma>\frac{1}{2}$. Thus $\psi(s)$ is analytic on $\sigma>\frac{1}{2}$. Moreover, since $L\left(2 s, \chi_{0}\right)$ has a pole at $s=\frac{1}{2}$ we have $\psi(s) \rightarrow 0$ as $s \rightarrow \frac{1}{2}$.

We assume temporarily that s is real and $s>1$. Then $\psi(s)$ has an infinite product expansion

$$
\begin{aligned}
\psi(s) & =\prod_{p}\left(1-\chi(p) p^{-s}\right)^{-1}\left(1-\chi_{0}(p) p^{-s}\right)^{-1}\left(1-\chi_{0}(p) p^{-2 s}\right) \\
& =\prod_{p \nmid m} \frac{\left(1-p^{-2 s}\right)}{\left(1-p^{-s}\right)\left(1-\chi(p) p^{-s}\right)} .
\end{aligned}
$$

If $\chi(p)=-1$ the p -factor is equal to 1 . Thus

$$
\psi(s)=\prod_{\chi(p)=1} \frac{1+p^{-s}}{1-p^{-s}}
$$

where the product is over all p such that $\chi(p)=1$. Now,

$$
\frac{1+p^{-s}}{1-p^{-s}}=\left(1+p^{-s}\right)\left(\sum_{k=0}^{\infty} p^{-k s}\right)=1+2 p^{-s}+2 p^{-2 s}+\cdots+.
$$

Applying Lemma 1.3 we find that $\psi(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ where $a_n\geq 0$ and the series converges for $s>1$. Note that $a_1=1$.

We once again consider $\psi(s)$ as a function of a complex variable and expand it in a power series about $s=2, \psi(s)=\sum_{m=0}^{\infty}b_m(s-2)^m$. Since $\psi(s)$ is analytic for $\sigma>\frac{1}{2}$ the radius of convergence of this power series is at least $\frac{3}{2}$. To compute the b_m we use Taylor's theorem, i.e., $b_m=\psi^{(m)}(2)/m\,!$ where $\psi^{(m)}(s)$ is the m th derivative of $\psi(s)$. Since $\psi(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ we find $\psi^{(m)}(2)=\sum_{n=1}^{\infty}a_n(-\ln n)^m n^{-2}=(-1)^m c_m$ with $c_m\geq 0$. Thus $\psi(s)=\sum_{n=0}^{\infty}c_m(2-s)^m$ with c_m nonnegative and $c_0=\psi(2)=\sum_{n=1}^{\infty}a_n n^{-2}\geq a_1=\bar{1}$. It follows that for real s in the interval $\left(\frac{1}{2},2\right)$ we have $\bar{\psi}(s)\geq 1$. This contradicts $\psi(s)\to 0$ as $s\to\frac{1}{2}$, and so $L(1,\chi)\neq 0$.

Suppose χ is a nontrivial Dirichlet character. We want to show $G(s,\chi)$ remains bounded as $s\to 1$ through real values $s>1$.

Since $L(1,\chi)\neq 0$ there is a disc D about $L(1,\chi)$ such $0\notin D$. Let $\ln z$ be a single-valued branch of the logarithm defined on D . There is a $\delta>0$ such that $L(s,\chi)\in D$ for $s\in(1,1+\delta)$. Consider $\ln L(s,\chi)$ and $G(s,\chi)$ for s in this interval. The exponential of both functions is $L(s,\chi)$. Thus there is an integer N such that $G(s,\chi)=2\pi iN+\ln L(s,\chi)$ for $s\in(1,1+\delta)$. This implies $\lim_{s\to 1}G(s,\chi)$ exists and is equal to $2\pi iN+\ln L(1,\chi)$. Since $G(s,\chi)$ has a limit as $s\to 1$ it clearly remains bounded.

1.5 Evaluating $L(s,\chi)$ at Negative Integers

In the last section we showed how to analytically continue $L(s,\chi)$ into the region $\{s\in\mathbb{C}\mid\sigma>0\}$ .Riemann showed how to analytically continue these functions to the whole complex plane. As noted earlier this fact has important consequences for number theory. For example, the values $L(1-k,\chi)$, where k is a positive integer, are closely related to the Bernoulli numbers. A knowledge of these numbers has deep connections with the theory of cyclotomic fields. We will analytically continue $L(s,\chi)$ and evaluate the numbers $L(1-k,\chi)$ following a method due to D. Goss.

Before beginning we need to discuss some properties of the Γ -function. This is defined by

$$\Gamma(s)=\int_0^{\infty}e^{-t}t^{s-1}dt \tag{i}$$

It is not hard to see that the integral converges and defines an analytic function on the region $\{s\in\mathbb{C}\mid\sigma>0\}$. For $\sigma>1$ we integrate by parts and find

$$\Gamma(s)=-e^{-t}t^{s-1}\Big|_0^{\infty}+(s-1)\int_0^{\infty}e^{-t}t^{s-2}dt$$

It follows that $\Gamma(s)=(s-1)\Gamma(s-1)$ for $\sigma>1$. Since $\Gamma(1)=\int_0^{\infty}e^{-t}dt=1$ we see $\Gamma(n+1)=n\,!$ for positive integers n .

The functional equation $\Gamma(s)=(s-1)\Gamma(s-1)$ enables us to analytically continue $\Gamma(s)$ by a step by step process. If $\sigma>-1$ we define $\Gamma_1(s)$ by

$$\Gamma_1(s)=\frac{1}{s}\Gamma(s+1) \tag{ii}$$

For $\sigma>0, \Gamma_1(s)=\Gamma(s)$. Moreover, $\Gamma_1(s)$ is analytic on $\sigma>-1$ except for a simple pole at $s=0$. Similarly,

Definition 1.3

if k is a positive integer

$$\Gamma_k(s)=\frac{1}{s(s+1)\cdots(s+k-1)}\Gamma(s+k).$$

♣

$\Gamma_k(s)$ is analytic on $\{s \in \mathbb{C} \mid \sigma > -k\}$ except for simple poles at $s = 0, -1, \ldots, 1-k$ and $\Gamma_k(s) = \Gamma(s)$ for $\sigma > 0$. These functions fit together to give an analytic continuation of $\Gamma(s)$ to the whole complex plane with poles at the nonpositive integers and nowhere else. From now on $\Gamma(s)$ will denote this extended function. We remark, without proof, that $\Gamma(s)^{-1}$ is entire.

We will now show how to analytically continue $\zeta(s)$ by the same process. It is necessary to express $\zeta(s)$ as an integral. In Equation (i) substitute nt for t . We find, for $\sigma > 1$

$$n^{-s}\Gamma(s) = \int_0^\infty e^{-nt} t^{s-1} dt \tag{iii}$$

Sum both sides of (iii) for $n = 1, 2, 3, \ldots$. It is not hard to justify interchanging the sum and the integral. The result is

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{e^{-t}}{1-e^{-t}} t^{s-1} dt. \tag{iv}$$

If we tried to integrate by parts at this stage we would be blocked by the fact that $1 - e^{-t}$ is zero when $t = 0$. To get around this we use a trick. In (iv) substitute $2t$ for t . We find

$$2^{1-s}\Gamma(s)\zeta(s) = 2\int_0^\infty \frac{e^{-2t}}{1-e^{-2t}} t^{s-1} dt. \tag{v}$$

Definition 1.4

$\zeta^*(s) = \left(1 - 2^{1-s}\right)\zeta(s)$ *and* $R(x) = x/(1-x) - 2\left(x^2/\left(1-x^2\right)\right)$. ♣

Subtracting (v) from (iv) yields

$$\Gamma(s)\zeta^*(s) = \int_0^\infty R\left(e^{-t}\right) t^{s-1} dt. \tag{vi}$$

What has been gained? A simple algebraic manipulation shows $R(x) = x/(1+x)$. Thus $R\left(e^{-t}\right) = e^{-t}/\left(1+e^{-t}\right)$ has a denominator that does not vanish at $t = 0$. The integral in Equation (vi) thus converges for $\sigma > 0$ and this equation provides a continuation for $\zeta(s)$ to the region $\{s \in \mathbb{C} \mid \sigma > 0\}$.

Let $R_0(t) = R\left(e^{-t}\right)$ and for $m \geq 1, R_m(t) = \left(d^m/dt^m\right) R\left(e^{-t}\right)$. It is easy to see that $R_m(t) = e^{-t} P_m\left(e^{-t}\right)\left(1+e^{-t}\right)^{-2^m}$ where P_m is a polynomial. It follows that $R_m(0)$ is finite and $R_m(t)/e^{-t}$ is bounded as $t \to \infty$. These facts enable us to repeatedly integrate by parts in Equation (vi).

Take $u = R\left(e^{-t}\right)$ and $dv = t^{s-1}dt$. Then $du = R_1(t)\,dt$ and $v = t^s/s$. Thus

$$\Gamma(s)\zeta^*(s) = \frac{1}{s} t^s R_0(t)\Big|_0^\infty - \frac{1}{s}\int_0^\infty R_1(t)\, t^s dt$$

and so

$$\Gamma(s+1)\zeta^*(s) = -\int_0^\infty R_1(t)\, t^s dt \tag{vii}$$

The integral in (vii) converges to an analytic function in $\{s \in \mathbb{C} \mid \sigma > -1\}$, and provides an analytic continuation of $\zeta(s)$ to this region. Continuing this process we find for k a positive integer

$$\Gamma(s+k)\zeta^*(s) = (-1)^k \int_0^\infty R_k(t)\, t^{s+k-1} dt \tag{viii}$$

where the integral converges to an analytic function of s for $\sigma > -k$. This procedure provides an analytic continuation of $\zeta(s)$ to the whole complex plane. We continue to use the notation $\zeta(s)$ for the extended function.

Proposition 1.10

Let k be a positive integer. Then, $\zeta(0) = -\frac{1}{2}$ and for $k > 1, \zeta(1-k) = -B_k/k$.

Assume now that χ is a nontrivial character modulo m . To handle $L(s,\chi)$ we proceed in exactly the same way as for $\zeta(s)$. In Equation (iii) multiply both sides by $\chi(n)$ and sum over n . The result is $\Gamma(s) L(s,\chi) = \int_0^\infty F_\chi(e^{-t}) t^{s-1} dt$, where

$$F_\chi(e^{-t}) = \sum_{n=1}^{\infty} \chi(n) e^{-nt} = \sum_{a=1}^{m} \chi(a) \sum_{k=0}^{\infty} e^{-(a+km)t} = \sum_{a=1}^{m} \chi(a) \frac{e^{-at}}{1-e^{-mt}}.$$

If we define $L^*(s,\chi) = \left(1-2^{1-s}\right) L(s,\chi)$, then in the same way as we derived Equation (vi) we find

$$\Gamma(s) L^*(s,\chi) = \int_0^\infty R_\chi(e^{-t}) t^{s-1} dt \tag{ix}$$

where

$$R_\chi(x) = F_\chi(x) - 2F_\chi(x^2)$$

$$= \sum_{a=1}^{m} \chi(a) \left(\frac{x^a}{1-x^m} - 2\frac{x^{2a}}{1-x^{2m}} \right)$$

$$= \sum_{a=1}^{m} \chi(a) x^a \left(\frac{1+x^m-2x^a}{(1-x)(1+x+\cdots+x^{2m-1})} \right).$$

For each value of a we see $x=1$ is a root of $1+x^m-2x^a$, and it follows that $R_\chi(x)$ has the form

$$R_\chi(x) = \frac{x f(x)}{1+x+\cdots+x^{2m-1}}$$

where $f(x)$ is a polynomial. Let $R_{\chi,0}(t) = R_\chi(e^{-t})$ and $R_{\chi,n} = (d^n/dt^n) R_\chi(e^{-t})$. By repeated integration by parts we find in the same way that we derived Equation (viii) that

$$\Gamma(s+k) L^*(s,\chi) = (-1)^k \int_0^\infty R_{\chi,k}(t) t^{s+k-1} dt. \tag{x}$$

The integral in (x) converges to an analytic function in $\{s \in \mathbb{C} \mid \sigma > -k\}$. These formulas provide an analytic continuation of $L^*(s,\chi)$ and thus $L(s,\chi)$ to the whole complex plane.

Before attempting to evaluate $L(s,\chi)$ at the negative integers we need a definition.

Definition 1.5

Let χ be a nontrivial Dirichlet character modulo m . The generalized Bernoulli number $B_{n,\chi}$ is defined by the following formula

$$\sum_{a=1}^{m} \chi(a) \frac{te^{at}}{e^{mt}-1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} t^n \tag{xi}$$

In the literature it is usual to define $B_{n,\chi}$ in this manner only if χ is a primitive character modulo m . We will discuss this point later.

Lemma 1.4

$tF_\chi(e^{-t}) = \sum_{n=0}^{\infty} (-1)^n (B_{n,\chi}/n!) t^n$.

Proposition 1.11

Let k be a positive integer. Then $L(1-k,\chi) = -B_{k,\chi}/k$.

Proof In Equation (x) substitute $s = 1-k$. The result is $\left(1-2^k\right) L(1-k,\chi) = (-1)^k \int_0^\infty R_{\chi,k}(t)\,dt$. Since $R_{\chi,k}(t) = (d/dt) R_{\chi,k-1}(t)$ it follows that $\left(1-2^k\right) L(1-k,\chi) = (-1)^{k-1} R_{\chi,k-1}(0)$. Since

$$R_{\chi,k-1}(t) = \frac{d^{k-1}}{dt^{k-1}} R_\chi\left(e^{-t}\right)$$

and $R_\chi\left(e^{-t}\right) = F_\chi\left(e^{-t}\right) - 2F_\chi\left(e^{-2t}\right) = (1/t) \sum_{k=1}^{\infty} (-1)^k \left(1-2^k\right)\left(B_{k,\chi}/k!\right) t^k$ (by Lemma 1.1) we see that $(-1)^{k-1} R_{\chi,k-1}(0) = -\left(1-2^k\right)\left(B_{k,\chi}/k\right)$. Thus, $L(1-k,\chi) = -B_{k,\chi}/k$ as asserted.

It follows from Equation (xi) that the numbers $B_{k,\chi}$ are in the field generated over $\mathbb{Q}$ by the values of χ . Thus, in particular, they are algebraic numbers.

As mentioned earlier it is usual to define $B_{n,\chi}$ by Equation (xi) only when χ is a primitive character modulo m . This means that χ when restricted to $\{n \in \mathbb{Z} \mid (n,m) = 1\}$ does not have a smaller period than m . The trivial character is primitive only for the modulus 1 . From Equation (xi) we then have

$$\sum_{n=0}^{\infty} \frac{B_{n,\chi_0}}{n!} t^n = \frac{te^t}{e^t-1} = t + \frac{t}{e^t-1} = 1 + \frac{1}{2}t + \sum_{n=2}^{\infty} \frac{B_n}{n!} t^n.$$

Thus $-B_{1,\chi_0} = B_1$ and $B_{n,\chi_0} = B_n$ for $n \neq 2$. It is in this sense that the $B_{n,\chi}$ are "generalized Bernoulli numbers."

Note

The $B_{n,\chi}$ have many interesting arithmetic properties. The interested reader should consult Iwasawa's monograph. This monograph is devoted to showing how the equation $L(1-k,\chi) = -B_{k,\chi}/k$ leads to p -adic L -functions and to the remarkable connection between these functions and the theory of cyclotomic fields.

Bibliography

[1] A Classical Introduction to Modern Number Theory Kenneth Ireland, Michael Rosen Graduate Texts in Mathematics

[2] Introduction to Analytic Number Theory Tom M. Apostol

[3] Analytic Number Theory Henryk Iwaniec: Rutgers University, Piscataway, NJ Emmanuel Kowalski : Université Bordeaux I, Talence, France