Mertens's Theorems

Author: Jiahai Wang

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Chapter 1 Mertens's theorems

In this section, we derive some important results about the distribution of prime numbers that were originally proved by Mertens.

Lemma 1.1

For any real number $x \geq 1$ we have

$$0 \leq \sum_{n \leq x} \log\left(\frac{x}{n}\right) < x$$

Proof It follows that

$$\sum_{1 \leq n \leq x} \log\left(\frac{x}{n}\right) < \log x + \int_1^x \log\left(\frac{x}{t}\right) dt$$

$$= x \log x - (x \log x - x + 1)$$

$$< x$$

The function $\Lambda(n)$, called von Mangoldt's function, is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ is a prime power} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\psi(x) = \sum_{1 \leq m \leq x} \Lambda(m).$$

Theorem 1.1 (Mertens)

For any real number $x \geq 1$, we have

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

Proof Let $N = [x]$. Then

$$0 \leq \sum_{n \leq x} \log \frac{x}{n} = N \log x - \sum_{n=1}^{N} \log n = x \log x - \log N! + O(\log x) < x$$

and so

$$\log N! = x \log x + O(x).$$

It follows that

$$\log N! = \sum_{p \leq N} v_p(N) \log p$$

$$= \sum_{p \leq N} \sum_{k=1}^{[\log N / \log p]} \left[\frac{N}{p^k}\right] \log p$$

$$
\begin{aligned}
&= \sum_{p^k \le N} \left[\frac{N}{p^k}\right] \log p \\
&= \sum_{p^k \le x} \left[\frac{x}{p^k}\right] \log p \\
&= \sum_{n \le x} \left[\frac{x}{n}\right] \Lambda(n) \\
&= \sum_{n \le x} \left(\frac{x}{n} + O(1)\right) \Lambda(n) \\
&= x \sum_{n \le x} \frac{\Lambda(n)}{n} + O\left(\sum_{n \le x} \Lambda(n)\right) \\
&= x \sum_{n \le x} \frac{\Lambda(n)}{n} + O(\psi(x)) \\
&= x \sum_{n \le x} \frac{\Lambda(n)}{n} + O(x).
\end{aligned}
$$

Therefore,

$$
x \sum_{n \le x} \frac{\Lambda(n)}{n} + O(x) = x \log x + O(x)
$$

and

$$
\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1)
$$

Theorem 1.2 (Mertens)

For any real number $x \ge 1$, we have

$$
\sum_{p \le x} \frac{\log p}{p} = \log x + O(1)
$$

♡

Proof Since

$$
\begin{aligned}
0 \le \sum_{n \le x} \frac{\Lambda(n)}{n} - \sum_{p \le x} \frac{\log p}{p}
&= \sum_{\substack{p^k \le x \\ k \ge 2}} \frac{\log p}{p^k} \\
&\le \sum_{p \le x} \log p \sum_{k=2}^{\infty} \frac{1}{p^k} \\
&\le \sum_{p \le x} \frac{\log p}{p(p-1)} \\
&\le 2 \sum_{p \le x} \frac{\log p}{p^2}
\end{aligned}
$$

$$\leq 2 \sum_{n=1}^{\infty} \frac{\log n}{n^2}$$

$$= O(1) \ ,$$

it follows from Theorem 1.1 that

$$\sum_{p \leq x} \frac{\log p}{p} = \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(1) = \log x + O(1) \, .$$

Theorem 1.3

There exists a constant $b_1 > 0$ such that

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + b_1 + O\left(\frac{1}{\log x}\right)$$

for $x \geq 2$.

♡

Proof We can write

$$\sum_{p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{\log p}{p} \frac{1}{\log p} = \sum_{n \leq x} u(n) f(n) \, ,$$

where

$$u(n) = \begin{cases} \frac{\log p}{p} & \text{if } n = p \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(t) = \frac{1}{\log t}.$$

We define the functions $U(t)$ and $g(t)$ by

$$U(t) = \sum_{n \leq t} u(n) = \sum_{p \leq t} \frac{\log p}{p} = \log t + g(t) \, .$$

Then $U(t) = 0$ for $t < 2$ and $g(t) = O(1)$ by Theorem 1.2. Therefore, the integral $\int_2^{\infty} g(t) / \left(t (\log t)^2\right) dt$ converges absolutely, and

$$\int_x^{\infty} \frac{g(t) \, dt}{t (\log t)^2} = O\left(\frac{1}{\log x}\right) .$$

Since $f(t)$ is continuous and $U(t)$ is increasing, we can express the sum $\sum\limits_{p \leq x} 1/p$ as a Riemann-Stieltjes integral. Note that $U(t) = 0$ for $t < 2$. By partial summation, we obtain

$$\sum_{p \leq x} \frac{1}{p} = \sum_{n \leq x} u(n) f(n)$$

$$= \frac{1}{2} + \int_2^x f(t) \, dU(t)$$

$$= f(x) U(x) - \int_2^x U(t) \, df(t)$$

$$= \frac{\log x + g(x)}{\log x} - \int_2^x U(t) f'(t)\, dt$$

$$= 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{\log t + g(t)}{t(\log t)^2} dt$$

$$= \int_2^x \frac{1}{t \log t} dt + \int_2^\infty \frac{g(t)}{t(\log t)^2} dt - \int_x^\infty \frac{g(t)}{t(\log t)^2} dt + 1 + O\left(\frac{1}{\log x}\right)$$

$$= \log\log x - \log\log 2 + \int_2^\infty \frac{g(t)}{t(\log t)^2} dt + 1 + O\left(\frac{1}{\log x}\right)$$

$$= \log\log x + b_1 + O\left(\frac{1}{\log x}\right)$$

where

$$b_1 = 1 - \log\log 2 + \int_2^\infty \frac{g(t)}{t(\log t)^2} dt \tag{1.1}$$

From the Taylor series for $\log(1-x)$, we see that

$$0 < \log\left(1 - \frac{1}{p}\right)^{-1} - \frac{1}{p} = \sum_{n=2}^{\infty} \frac{1}{np^n} < \sum_{n=2}^{\infty} \frac{1}{p^n} = \frac{1}{p(p-1)}.$$

It follows from the comparison test that the series

$$b_2 = \sum_p \left(\log\left(1 - \frac{1}{p}\right)^{-1} - \frac{1}{p}\right) = \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k} \tag{1.2}$$

converges.

Lemma 1.2

Let b_1 and b_2 be the positive numbers defined by (1.1) and (1.2). Then

$$b_1 + b_2 = \gamma$$

where γ is Euler's constant.

Proof Let $0 < \sigma < 1$. We define the function $F(\sigma)$ by

$$F(\sigma) = \log\zeta(1+\sigma) - \sum_p \frac{1}{p^{1+\sigma}}$$

$$= \sum_p \left(\log\left(1 - \frac{1}{p^{1+\sigma}}\right)^{-1} - \frac{1}{p^{1+\sigma}}\right)$$

$$= \sum_p \sum_{n=2}^{\infty} \frac{1}{np^{n(1+\sigma)}}.$$

By the Weierstrass M-test, the last series converges uniformly for $\sigma \geq 0$ and so represents a continuous function for $\sigma \geq 0$. Therefore,

$$\lim_{\sigma \to 0^+} F(\sigma) = b_2$$

We shall find alternative representations for the functions $\log\zeta(1+\sigma)$ and $\sum_p p^{-1-\sigma}$. Since

$$1-\sigma+\frac{\sigma^2}{2e}<e^{-\sigma}<1-\sigma+\frac{\sigma^2}{2}$$

for $0<\sigma<1$, it follows that

$$1-\frac{\sigma}{2}<\frac{1-e^{-\sigma}}{\sigma}<1-\frac{\sigma}{2e}$$

and

$$1+\frac{\sigma}{2e}<1+\frac{\sigma}{2e-\sigma}<\frac{\sigma}{1-e^{-\sigma}}<1+\frac{\sigma}{2-\sigma}<1+\sigma.$$

Therefore,

$$0<\log\sigma+\log\left(1-e^{-\sigma}\right)^{-1}<\sigma,$$

and so

$$\log\frac{1}{\sigma}=\log\left(1-e^{-\sigma}\right)^{-1}+O\left(\sigma\right).$$

we have

$$\log\zeta\left(1+\sigma\right)=\log\frac{1}{\sigma}+O\left(\sigma\right)$$

$$=\log\left(1-e^{-\sigma}\right)^{-1}+O\left(\sigma\right)$$

$$=\sum_{n=1}^{\infty}\frac{e^{-\sigma n}}{n}+O\left(\sigma\right).$$

As we know

$$L\left(x\right)=\sum_{n\leq x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right)$$

for $x\geq 1$. Let $f\left(x\right)=e^{-\sigma x}$. By partial summation, we have

$$\log\zeta\left(1+\sigma\right)=\sum_{n=1}^{\infty}\frac{f\left(n\right)}{n}+O\left(\sigma\right)$$

$$=\int_0^{\infty}f\left(x\right)dL\left(x\right)+O\left(\sigma\right)$$

$$=-\int_0^{\infty}L\left(x\right)df\left(x\right)+O\left(\sigma\right)$$

$$=\sigma\int_0^{\infty}e^{-\sigma x}L\left(x\right)dx+O\left(\sigma\right).$$

By Theorem 1.3,

$$S\left(x\right)=\sum_{p\leq x}\frac{1}{p}=\log\log x+b_1+O\left(\frac{1}{\log x}\right)$$

for $x\geq 2$. Let $g\left(x\right)=x^{-\sigma}$. Again, by partial summation we have

$$\sum_{p}\frac{1}{p^{1+\sigma}}=\sum_{p}\frac{g\left(p\right)}{p}=\int_1^{\infty}g\left(x\right)dS\left(x\right)=-\int_1^{\infty}S\left(x\right)dg\left(x\right)$$

$$= \sigma \int_1^{\infty} \frac{S(x)\,dx}{x^{1+\sigma}}$$

$$= \sigma \int_0^{\infty} e^{-\sigma x} S(e^x)\,dx.$$

Since

$$S(e^x) = \log x + b_1 + O\left(\frac{1}{x}\right)$$

and

$$L(x) = \log x + \gamma + O\left(\frac{1}{x}\right),$$

it follows that

$$L(x) - S(e^x) = \gamma - b_1 + O\left(\frac{1}{x}\right) = \gamma - b_1 + O\left(\frac{1}{x+1}\right)$$

for $x \geq 1$. We also have

$$L(x) - S(e^x) = \gamma - b_1 + O\left(\frac{1}{x+1}\right)$$

for $0 \leq x \leq 1$. Therefore,

$$F(\sigma) = \log \zeta(1+\sigma) - \sum_p \frac{1}{p^{1+\sigma}}$$

$$= \sigma \int_0^{\infty} e^{-\sigma x} (L(x) - S(e^x))\,dx + O(\sigma)$$

$$= \sigma \int_0^{\infty} e^{-\sigma x} \left(\gamma - b_1 + O\left(\frac{1}{x+1}\right)\right) dx + O(\sigma)$$

$$= (\gamma - b_1)\,\sigma \int_0^{\infty} e^{-\sigma x} dx + O\left(\sigma \int_0^{\infty} \frac{e^{-\sigma x} dx}{x+1}\right) + O(\sigma)$$

$$= \gamma - b_1 + O\left(\sigma \int_0^{\infty} \frac{e^{-\sigma x} dx}{x+1}\right) + O(\sigma).$$

Since

$$\int_0^{\infty} \frac{e^{-\sigma x} dx}{x+1} < \int_0^{1/\sigma} \frac{e^{-\sigma x} dx}{x+1} + \int_{1/\sigma}^{\infty} \frac{e^{-\sigma x} dx}{x}$$

$$< \int_0^{1/\sigma} \frac{dx}{x+1} + \int_1^{\infty} \frac{e^{-y} dy}{y}$$

$$= \log\left(\frac{1}{\sigma} + 1\right) + O(1)$$

$$\ll \log\left(\frac{1}{\sigma} + 1\right)$$

It follows that

$$F(\sigma) = \gamma - b_1 + O\left(\sigma \log\left(\frac{1}{\sigma} + 1\right)\right).$$

we have

$$b_2 = \lim_{\sigma \to 0^+} F(\sigma) = \gamma - b_1.$$

Theorem 1.4 (Mertens's formula)

For $x \geq 2$,

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^{\gamma} \log x + O(1)$$

where γ is Euler's constant.

Proof We begin with two observations. First,

$$\begin{aligned}
\sum_{p>x} \sum_{k=2}^{\infty} \frac{1}{kp^k} &< \sum_{p>x} \frac{1}{p(p-1)} \\
&< \sum_{n>x} \frac{1}{n(n-1)} \\
&= \sum_{n>x} \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= O\left(\frac{1}{x}\right) \\
&= O\left(\frac{1}{\log x}\right).
\end{aligned}$$

Second, since $\exp(t) = 1 + O(t)$ for t in any bounded interval and $O(1/\log x)$ is bounded for $x \geq 2$, it follows that

$$\exp\left(O\left(\frac{1}{\log x}\right)\right) = 1 + O\left(\frac{1}{\log x}\right).$$

Therefore,

$$\begin{aligned}
\log \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} &= \sum_{p \leq x} \log\left(1 - \frac{1}{p}\right)^{-1} \\
&= \sum_{p \leq x} \sum_{k=1}^{\infty} \frac{1}{kp^k} \\
&= \sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} \sum_{k=2}^{\infty} \frac{1}{kp^k} \\
&= \log\log x + b_1 + O\left(\frac{1}{\log x}\right) + b_2 - \sum_{p>x} \sum_{k=2}^{\infty} \frac{1}{kp^k} \\
&= \log\log x + \gamma + O\left(\frac{1}{\log x}\right)
\end{aligned}$$

since $b_1 + b_2 = \gamma$ by Lemma 1.2, and so

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^{\gamma} \log x \exp\left(O\left(\frac{1}{\log x}\right)\right)$$

$$= e^{\gamma} \log x \left(1 + O\left(\frac{1}{\log x}\right)\right)$$

$$= e^{\gamma} \log x + O(1) \ .$$

This is Mertens's formula.

The following result will be used in the proof of Chen's theorem.

Theorem 1.5

For any $\varepsilon > 0$, there exists a number $u_1 = u_1(\varepsilon)$ such that

$$\prod_{u \leq p < z} \left(1 - \frac{1}{p}\right)^{-1} < (1 + \varepsilon) \frac{\log z}{\log u}$$

for any $u_1 \leq u < z$. ♡

Proof Let γ be Euler's constant, and choose $\delta > 0$ such that

$$\frac{\gamma + \delta}{\gamma - \delta} < 1 + \varepsilon$$

By Theorem 1.4, we have

$$\prod_{p < x} \left(1 - \frac{1}{p}\right)^{-1} \sim \gamma \log x$$

and so there exists a number u_1 such that

$$(\gamma - \delta) \log x < \prod_{p < x} \left(1 - \frac{1}{p}\right)^{-1} < (\gamma + \delta) \log x$$

for all $x \geq u_1$. Therefore, if $u_1 \leq u < z$, we have

$$\prod_{u \leq p < z} \left(1 - \frac{1}{p}\right)^{-1} = \frac{\prod_{p < z} \left(1 - \frac{1}{p}\right)^{-1}}{\prod_{p < u} \left(1 - \frac{1}{p}\right)^{-1}}$$

$$< \frac{(\gamma + \delta) \log z}{(\gamma - \delta) \log u}$$

$$< (1 + \varepsilon) \frac{\log z}{\log u}.$$